49. Multiple Choice Two boats start at the same point and speed away along courses that form a 110° angle. If one boat travels at 24 miles per hour and the other boat travels at 32 miles per hour, how far apart are the boats after 30 minutes?

(A) 21 miles **(B)** 22 miles **(C)** 23 miles
(D) 24 miles **(E)** 25 miles

50. Multiple Choice What is the measure of the smallest angle in a triangle with sides 12, 17, and 25?

(A) 21° **(B)** 22° **(C)** 23° **(D)** 24° **(E)** 25°

Explorations

51. Find the area of a regular polygon with n sides inscribed inside a circle of radius r. (Express your answer in terms of n and r.)

52. (a) Prove the identity: $\dfrac{\cos A}{a} = \dfrac{b^2 + c^2 - a^2}{2abc}$.

(b) Prove the (tougher) identity:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

[*Hint:* use the identity in part (a), along with its other variations.]

53. Navigation Two ships leave a common port at 8:00 A.M. and travel at a constant rate of speed. Each ship keeps a log showing its distance from port and its distance from the other ship. Portions of the logs from later that morning for both ships are shown in the following tables.

Time	Naut mi from port	Naut mi from ship B
9:00	15.1	8.7
10:00	30.2	17.3

Time	Naut mi from port	Naut mi from ship A
9:00	12.4	8.7
11:00	37.2	26.0

(a) How fast is each ship traveling? (Express your answer in knots, which are nautical miles per hour.)

(b) What is the angle of intersection of the courses of the two ships?

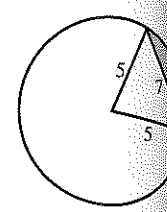
(c) How far apart are the ships at 12:00 noon if they maintain the same courses and speeds?

Extending the Ideas

54. Prove that the area of a triangle can be found with the formula

$$\triangle \text{ Area} = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

55. A **segment** of a circle is the region enclosed between a chord of a circle and the arc intercepted by the chord. Find the area of a segment intercepted by a 7-inch chord in a circle of radius 5 inches.



Math at Work

I got into medicine because I always liked a good challenge. Medicine can be like solving a puzzle, which I enjoy. I chose anesthesiology because it's even more challenging than other fields of medicine. Any surgical specialty tends to offer more difficult problems than, say, treating the sniffles.

What I enjoy most about my job is trying to gain people's confidence in a 5 minute interview. I can tell that some people will accept my judgments after our first interview, and others will question everything I do.

One good example of how we use math in medicine is when a patient goes into shock. Typically, the patient's blood pressure bottoms out, and his natural mechanisms are unable to raise it. We give the patient dopamine to raise his blood pressure, but we need to make sure there is a consistent level of dopamine entering the bloodstream. For instance, if we have 400 mg of dopamine mixed into 250 cc of saline, we need to calculate how fast the intravenous drip should be so that there is 5 mg of dopamine in the bloodstream per kilo of body weight. Since all patients have different body weights, all patients require different rates of intravenous drip.

Ernest Newkirk, M.D.,

CHAPTER 5 Key Ideas

PROPERTIES, THEOREMS, AND FORMULAS

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CHAPTER 5 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1 and 2, write the expression as the sine, cosine, or tangent of an angle.

1. $2 \sin 100° \cos 100°$ **2.** $\dfrac{2 \tan 40°}{1 - \tan^2 40°}$

In Exercises 3 and 4, simplify the expression to a single term. Support your answer graphically.

3. $(1 - 2 \sin^2 \theta)^2 + 4 \sin^2 \theta \cos^2 \theta$

4. $1 - 4 \sin^2 x \cos^2 x$

In Exercises 5–22, prove the identity.

5. $\cos 3x = 4 \cos^3 x - 3 \cos x$

6. $\cos^2 2x - \cos^2 x = \sin^2 x - \sin^2 2x$

7. $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

8. $2 \sin \theta \cos^3 \theta + 2 \sin^3 \theta \cos \theta = \sin 2\theta$

9. $\csc x - \cos x \cot x = \sin x$

10. $\dfrac{\tan \theta + \sin \theta}{2 \tan \theta} = \cos^2 \left(\dfrac{\theta}{2}\right)$

11. $\dfrac{1 + \tan \theta}{1 - \tan \theta} + \dfrac{1 + \cot \theta}{1 - \cot \theta} = 0$

12. $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$

13. $\cos^2 \left(\dfrac{t}{2}\right) = \dfrac{1 + \sec t}{2 \sec t}$

14. $\dfrac{\tan^3 \gamma - \cot^3 \gamma}{\tan^2 \gamma + \csc^2 \gamma} = \tan \gamma - \cot \gamma$

15. $\dfrac{\cos \phi}{1 - \tan \phi} + \dfrac{\sin \phi}{1 - \cot \phi} = \cos \phi + \sin \phi$

16. $\dfrac{\cos(-z)}{\sec(-z) + \tan(-z)} = 1 + \sin z$

17. $\sqrt{\dfrac{1 - \cos y}{1 + \cos y}} = \dfrac{1 - \cos y}{|\sin y|}$ **18.** $\sqrt{\dfrac{1 - \sin \gamma}{1 + \sin \gamma}} = \dfrac{|\cos \gamma|}{1 + \sin \gamma}$

19. $\tan\left(u + \dfrac{3\pi}{4}\right) = \dfrac{\tan u - 1}{1 + \tan u}$

20. $\dfrac{1}{4} \sin 4\gamma = \sin \gamma \cos^3 \gamma - \cos \gamma \sin^3 \gamma$

21. $\tan \dfrac{1}{2}\beta = \csc \beta - \cot \beta$

22. $\arctan t = \dfrac{1}{2} \arctan \dfrac{2t}{1 - t^2}, \quad -1 < t < 1$

In Exercises 23 and 24, use a grapher to conjecture whether the equation is likely to be an identity. Confirm your conjecture.

23. $\sec x - \sin x \tan x = \cos x$

24. $(\sin^2 \alpha - \cos^2 \alpha)(\tan^2 \alpha + 1) = \tan^2 \alpha - 1$

In Exercises 25–28, write the expression in terms of $\sin x$ and $\cos x$ only.

25. $\sin 3x + \cos 3x$ **26.** $\sin 2x + \cos 3x$

27. $\cos^2 2x - \sin 2x$ **28.** $\sin 3x - 3 \sin 2x$

In Exercises 29–34, find the general solution without using a calculator. Give exact answers.

29. $\sin 2x = 0.5$ **30.** $\cos x = \dfrac{\sqrt{3}}{2}$

31. $\tan x = -1$ **32.** $2 \sin^{-1} x = \sqrt{2}$

33. $\tan^{-1} x = 1$ **34.** $2 \cos 2x = 1$

In Exercises 35–38, solve the equation graphically. Find all solutions in the interval $[0, 2\pi)$.

35. $\sin^2 x - 3 \cos x = -0.5$

36. $\cos^3 x - 2 \sin x - 0.7 = 0$

37. $\sin^4 x + x^2 = 2$

38. $\sin 2x = x^3 - 5x^2 + 5x + 1$

In Exercises 39–44, find all solutions in the interval $[0, 2\pi)$ without using a calculator. Give exact answers.

39. $2 \cos x = 1$

40. $\sin 3x = \sin x$

41. $\sin^2 x - 2 \sin x - 3 = 0$

42. $\cos 2t = \cos t$

43. $\sin (\cos x) = 1$

44. $\cos 2x + 5 \cos x = 2$

In Exercises 45–48, solve the inequality. Use any method, but give exact answers.

45. $2 \cos 2x > 1$ for $0 \le x < 2\pi$

46. $\sin 2x > 2 \cos x$ for $0 < x \le 2\pi$

47. $2 \cos x < 1$ for $0 \le x < 2\pi$

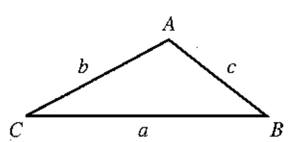
48. $\tan x < \sin x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

In Exercises 49 and 50, find an equivalent equation of the form $y = a \sin (bx + c)$. Support your work graphically.

49. $y = 3 \sin 3x + 4 \cos 3x$

50. $y = 5 \sin 2x - 12 \cos 2x$

In Exercises 51–58, solve $\triangle ABC$.



51. $A = 79°,\ B = 33°,\ a = 7$

52. $a = 5,\ b = 8,\ B = 110°$

53. $a = 8,\ b = 3,\ B = 30°$

54. $a = 14.7,\ A = 29.3°,\ C = 33°$

55. $A = 34°,\ B = 74°,\ c = 5$

56. $c = 41,\ A = 22.9°,\ C = 55.1°$

57. $a = 5,\ b = 7,\ c = 6$

58. $A = 85°,\ a = 6,\ b = 4$

In Exercises 59 and 60, find the area of $\triangle ABC$.

59. $a = 3,\ b = 5,\ c = 6$

60. $a = 10,\ b = 6,\ C = 50°$

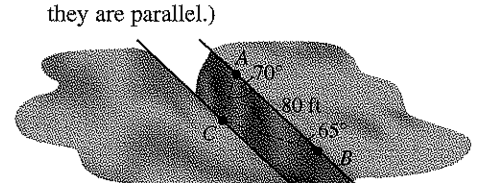
61. If $a = 12$ and $B = 28°$, determine the values of b that will produce the indicated number of triangles:

(a) Two **(b)** One **(c)** Zero

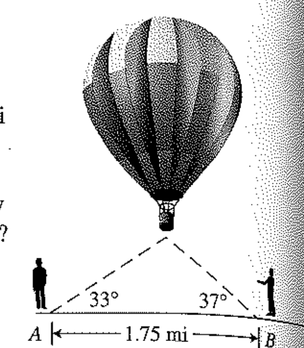
62. Surveying a Canyon Two markers A and B on the same side of a canyon rim are 80 ft apart, as shown in the figure. A hiker is located across the rim at point C. A surveyor determines that $\angle BAC = 70°$ and $\angle ABC = 65°$.

(a) What is the distance between the hiker and point A?

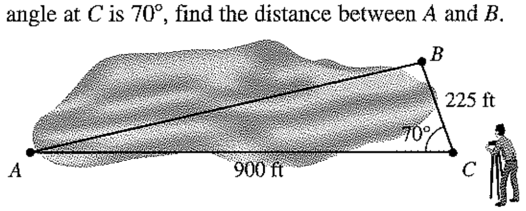
(b) What is the distance between the two canyon rims? (Assume they are parallel.)



63. Altitude A hot-air balloon is seen over Tucson, Arizona, simultaneously by two observers at points A and B that are 1.75 mi apart on level ground and in line with the balloon. The angles of elevation are as shown here. How high above ground is the balloon?



64. Finding Distance In order to determine the distance between two points A and B on opposite sides of a lake, a surveyor chooses a point C that is 900 ft from A and 225 ft from B, as shown in the figure. If the measure of the angle at C is 70°, find the distance between A and B.



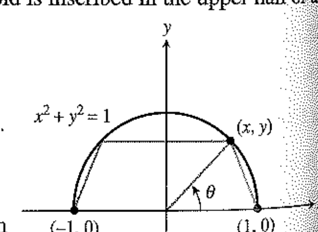
65. Finding Radian Measure Find the radian measure of the largest angle of the triangle whose sides have lengths 8, 9, and 10.

66. Finding a Parallelogram A parallelogram has sides of 15 and 24 ft, and an angle of 40°. Find the diagonals.

67. Maximizing Area A trapezoid is inscribed in the upper half of a unit circle, as shown in the figure.

(a) Write the area of the trapezoid as a function of θ.

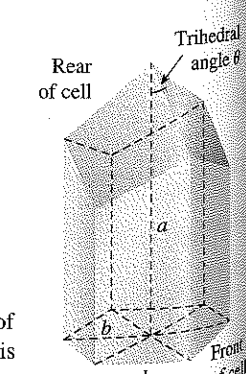
(b) Find the value of θ that maximizes the area of the trapezoid and the maximum area.



68. Beehive Cells A single cell in a beehive is a regular hexagonal prism open at the front with a trihedral cut at the back. Trihedral refers to a vertex formed by three faces of , a polyhedron. It can be shown that the surface area of a cell is given by

$$S(\theta) = 6ab + \frac{3}{2}b^2\left(-\cot \theta + \frac{\sqrt{3}}{\sin \theta}\right),$$

where θ is the angle between the axis of the prism and one of the back faces, a is the depth of the prism, and b is the length of the hexagonal front. Assume $a = 1.75$ in. and $b = 0.65$ in.

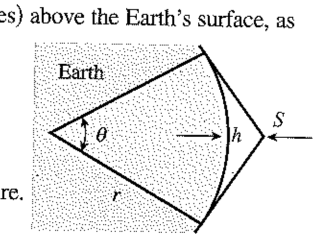


(a) Graph the function $y = S(\theta)$

(b) What value of θ gives the minimum surface area? (*Note:* This answer is quite close to the observed angle in nature.)

(c) What is the minimum surface area?

69. Cable Television Coverage A cable broadcast satellite S orbits a planet at a height h (in miles) above the Earth's surface, as shown in the figure. The two lines from S are tangent to the Earth's surface. The part of the Earth's surface that is in the broadcast area of the satellite is determined by the central angle θ indicated in the figure.



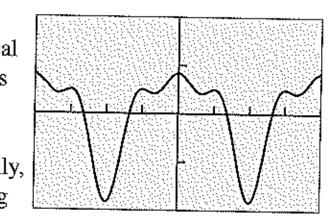
(a) Assuming that the Earth is spherical with a radius of 4000 mi, write h as a function of θ.

(b) Approximate θ for a satellite 200 mi above the surface of the Earth.

70. Finding Extremum Values The graph of

$$y = \cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$$

is shown in the figure. The x-values that correspond to local maximum and minimum points are solutions of the equation $\sin x - \sin 2x + \sin 3x = 0$. Solve this equation algebraically, and support your solution using the graph of y.



$[-2\pi, 2\pi]$ by $[-2, 2]$

71. Using Trigonometry in Geometry A regular hexagon whose sides are 16 cm is inscribed in a circle. Find the area inside the circle and outside the hexagon.

72. Using Trigonometry in Geometry A circle is inscribed in a regular pentagon whose sides are 12 cm. Find the area inside the pentagon and outside the circle.

73. Using Trigonometry in Geometry A wheel of cheese in the shape of a right circular cylinder is 18 cm in diameter and 5 cm thick. If a wedge of cheese with a central angle of 15° is cut from the wheel, find the volume of the cheese wedge.

74. Product-to-Sum Formulas Prove the following identities, which are called the **product-to-sum formulas**.

(a) $\sin u \sin v = \dfrac{1}{2}(\cos (u - v) - \cos (u + v))$

(b) $\cos u \cos v = \dfrac{1}{2}(\cos (u - v) + \cos (u + v))$

(c) $\sin u \cos v = \dfrac{1}{2}(\sin (u + v) + \sin (u - v))$

75. Sum-to-Product Formulas Use the product-to-sum formulas in Exercise 74 to prove the following identities, which are called the **sum-to-product formulas**.

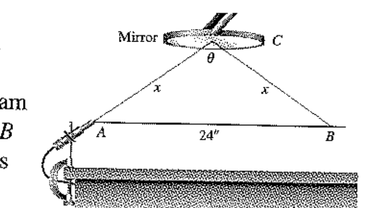
(a) $\sin u + \sin v = 2 \sin \dfrac{u+v}{2} \cos \dfrac{u-v}{2}$

(b) $\sin u - \sin v = 2 \sin \dfrac{u-v}{2} \cos \dfrac{u+v}{2}$

(c) $\cos u + \cos v = 2 \cos \dfrac{u+v}{2} \cos \dfrac{u-v}{2}$

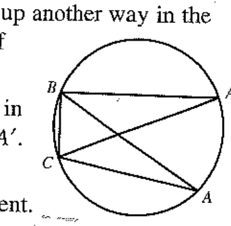
(d) $\cos u - \cos v = -2 \sin \dfrac{u+v}{2} \sin \dfrac{u-v}{2}$

76. Catching Students Faking Data Carmen and Pat both need to make up a missed physics lab. They are to measure the total distance $(2x)$ traveled by a beam of light from point A to point B and record it in 20° increments of θ as they adjust the mirror (C) upward vertically. They report the following measurements. However, only one of the students actually did the lab; the other skipped it and faked the data. Who faked the data, and how can you tell?



CARMEN		PAT	
θ	$2x$	θ	$2x$
160°	24.4″	160°	24.5″
140°	25.6″	140°	25.2″
120°	28.0″	120°	26.4″
100°	31.2″	100°	30.4″
80°	37.6″	80°	35.2″
60°	48.0″	60°	48.0″
40°	70.4″	40°	84.0″
20°	138.4″	20°	138.4″

77. An Interesting Fact about (sin A)/a The ratio $(\sin A)/a$ that shows up in the Law of Sines shows up another way in the geometry of $\triangle ABC$: It is the reciprocal of the radius of the circumscribed circle.



(a) Let $\triangle ABC$ be circumscribed as shown in the diagram, and construct diameter CA'. Explain why $\angle A'BC$ is a right angle.

(b) Explain why $\angle A'$ and $\angle A$ are congruent.

(c) If a, b, and c are the sides opposite angles A, B, and C as usual, explain why $\sin A' = a/d$, where d is the diameter of the circle.

(d) Finally, explain why $(\sin A)/a = 1/d$.

(e) Do $(\sin B)/b$ and $(\sin C)/c$ also equal $1/d$? Why?